QUANTUM COMPUTING FOR STUDENTS

MIRZA NUR HIDAYAT

Quantum Computing

for Students

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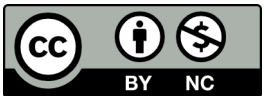

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Introduction

Welcome to Quantum Computing!

This is a quantum computing book for students. The book explains the very basics of quantum computing with examples and exercises, simply and clearly.

Happy reading,

Mirza Nur Hidayat

1

Qubit

Qubit, vector, superposition, probability, normalisation, matrix, Bloch sphere, quantum circuit

Qubit is the basic building block of quantum computing. It stands for quantum bit. Qubit can represent many combinations of 0 and 1 at the same time and is known as superposition.

Qubit 0 is written as $|0\rangle$, read as null vector or null ket, and qubit 1 as $|1\rangle$.

The superposition can be symbolised as $|\psi\rangle$, read as psi vector or psi state, and it has the equation

$$|\psi\rangle = a|0\rangle + b|1\rangle \tag{1.1}$$

The absolute squared value of a or written as $|a|^2$ is the probability of finding $|\psi\rangle$ in the vector $|0\rangle$. Likewise b, the value of $|b|^2$ is the probability of finding $|\psi\rangle$ in the vector $|1\rangle$.

The values of a and b have the following rule

$$|a|^2 + |b|^2 = 1 \tag{1.2}$$

The rule in equation (1.2) is known as the normalisation.

▹ Example 1.1.
Given a vector $|\psi_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$. Tell us about this superposition vector.

The above vector has value $\alpha = \frac{\sqrt{3}}{2}$ and value $b = \frac{1}{2}$ so that $|\alpha|^2 + |b|^2 = \frac{3}{4} + \frac{1}{4} = 1$. This value satisfies the normalisation rule as in (1.2). The probability at qubit 0 is $\frac{3}{4}$ and at qubit 1 is $\frac{1}{4}$. ◃

▹ Example 1.2.
Given a vector $|\psi_2\rangle = \frac{\sqrt{2}}{2}|0\rangle + \frac{1}{2}|1\rangle$. Describe this vector?

The above vector has values $a = \frac{\sqrt{2}}{2}$ and $b = \frac{1}{2}$ so that $|a|^2 + |b|^2 = \frac{2}{4} + \frac{1}{4} = \frac{3}{4} \neq 1$. This value does not satisfy the normalisation rule as in (1.2) so it is not a superposition vector and cannot be used in quantum computing. ◂

Next, the vector $|0\rangle$ can be written in matrix form with two rows and one column dimensioned, namely

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.3}$$

and the vector $|1\rangle$ as

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.4}$$

From (1.3) and (1.4), we can also write

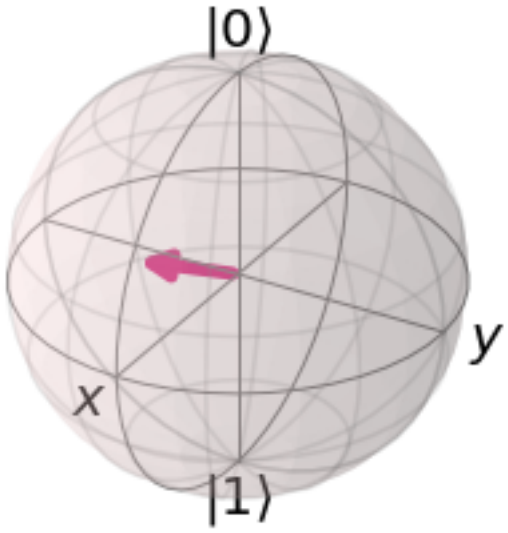

Figure 1.1: Bloch sphere of vector $|\psi_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

the vector representation of $|\psi\rangle$ on (1.1), namely

$$|\psi\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \qquad (1.5)$$

The next topic is Bloch sphere. A qubit in quantum coding can be visualised as a Bloch sphere.

The Bloch sphere in Figure 1.1 is a visu-

alisation of the vector $|\psi_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$. You can see the pink arrow in the sphere indicating the direction of the vector. The direction of this vector is between vector $|0\rangle$ and vector $|1\rangle$ and tends to point towards vector $|0\rangle$.

Note that the vector $|0\rangle$ has the direction of the arrow pointing exactly up or north pole and the vector $|1\rangle$ pointing exactly down or south pole.

Lastly in this chapter is quantum circuit. Quantum computing in its implementation can be presented in the form of circuit and is known as quantum circuit.

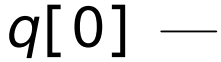

Figure 1.2: Quantum circuit

Figure 1.2 shows a quantum circuit. There is a qubit $q[0]$ and a horizontal line to its right.

Qubit $q[0]$ is the first qubit of a quantum circuit. In general, quantum computing starts with qubit $q[0]$ and not qubit $q[1]$. Qubit $q[0]$ has a vector $|0\rangle$.

Meanwhile, the horizontal line on the right is where to put the other components of the quantum circuit as will be presented in the following chapters.

- Exercise

1.1. Given a vector $|\phi\rangle = \frac{2}{3}|0\rangle + \frac{\sqrt{5}}{3}|1\rangle$. Define this vector.

1.2. Repeat Exercise 1.1. for a vector $|\varphi\rangle = \frac{1}{2}|0\rangle + |1\rangle$.

1.3. Determine the matrix representation of the vector in Exercise 1.1.

2

X Gate

Operator, X gate, matrix multiplication

Operator is a mathematical rule. When applied to a vector, it changes the vector into another vector.

In quantum computing, operator is known as quantum gate. One example of fundamental quantum gates is the *X* gate.

The X gate has a matrix of order two rows and two columns, namely

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

If the X gate is applied to a vector, for example the vector $|0\rangle$, it can be written as $X|0\rangle$.

Then what is the result of $X|0\rangle$? This question can be answered by matrix multiplication, i.e.

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2}$$

From (2.2) it is seen that the application of X gate on vector $|0\rangle$ will convert it to vector $|1\rangle$ or written $X|0\rangle = |1\rangle$.

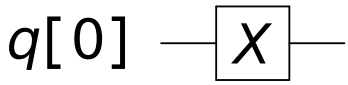

Figure 2.1: Quantum circuit of $X|0\rangle$

The quantum circuit of the $X|0\rangle$ operation is presented in Figure 2.1. There are two blocks or components namely $q[0]$ and a block with an X sign in the centre.

The component $q[0]$ as explained in the previous chapter is a qubit and has a vector $|0\rangle$.

Meanwhile, the block with an X sign in the centre is the X gate.

- Exercise

2.1. Given the operation of $X|1\rangle$. How is the matrix calculation? What is the result of the operation?

2.2. Given the action of $XX|0\rangle$. Determine the matrix calculation and the result of the action.

2.3. Sketch the quantum circuit of the operation $XX|0\rangle$.

3

H Gate

H gate, vector $|+\rangle$, *vector* $|-\rangle$

Just like the *X* gate, the Hadamard gate or *H* gate is also one of the fundamental gates in quantum computing.

The *H* gate changes the vector $|0\rangle$ into the vector $|+\rangle$ as well as converts the vector $|1\rangle$ into the vector $|-\rangle$.

What are the vector $|+\rangle$ and the vector $|-\rangle$? Here is the explanation.

The H gate has the following matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.1}$$

If the H gate is applied to the vector $|0\rangle$ or written as $H|0\rangle$ then the matrix calculation is

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.2}$$

The final value in (3.2) can be written in the form of

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3}$$

so

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \tag{3.4}$$

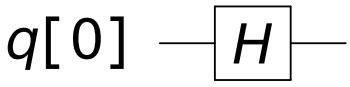

Figure 3.1: Quantum circuit of $H|0\rangle$

The value in (3.4) is known as the vector $|+\rangle$. Thus, the vector $|+\rangle$ can be written as

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad (3.5)$$

The quantum circuit of the $H|0\rangle$ operation is presented in Figure 3.1. There are qubit $q[0]$ and H gate in the quantum circuit.

The Bloch visualisation of the vector $|+\rangle$ is shown in Figure 3.2.

In the Bloch sphere in Figure 3.2 there is

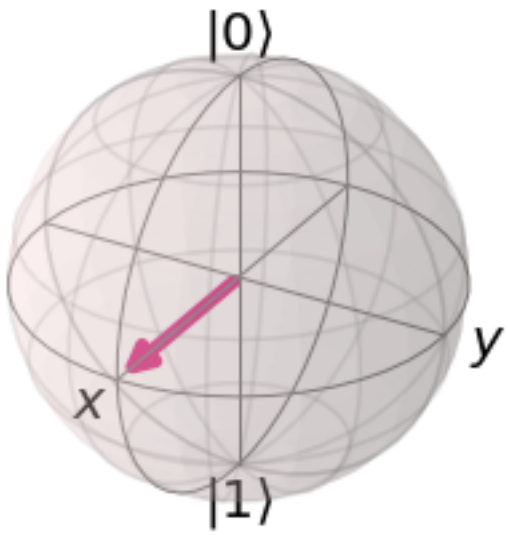

Figure 3.2: Bloch sphere of the vector |+⟩

an arrow pointing forwards or positive x-axis. This direction is exactly in the centre of the vector |0⟩ and the vector |1⟩.

- Exercise

3.1. The $H|1\rangle$ operation results in the vector $|-\rangle$. What is the matrix operation? Tell

me what the vector $|-\rangle$ is.

3.2. The vector $|-\rangle$ has the opposite direction to the vector $|+\rangle$. Sketch the Bloch sphere of vector $|-\rangle$.

3.3. Given an operation of $HH|0\rangle$. What is the result of this operation?

4

2-Qubit System

2-qubit system, tensor product

The discussion of quantum computing in the previous three chapters is a 1-qubit quantum system. This 1-qubit quantum system involves one qubit, $q[0]$.

In this fourth chapter, a discussion of the 2-qubit quantum system will be presented.

The 2-qubit quantum system or hereinafter written as the 2-qubit system, involves two qubits namely $q[0]$ and $q[1]$. Qubit $q[0]$ is the first qubit and $q[1]$ is the second qubit.

The 2-qubit system has a vector equation

$$|\psi\rangle = c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle \quad (4.1)$$

where $|c|^2$, $|d|^2$, $|e|^2$, and $|f|^2$ are the probability of finding $|\psi\rangle$ in the vectors $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, respectively.

The values of c, d, e, and f satisfy the rule of

$$|c|^2 + |d|^2 + |e|^2 + |f|^2 = 1 \quad (4.2)$$

Take a look at some examples below.

▹ Example 4.1.
Given a vector $|\psi_e\rangle = |10\rangle$. Describe this vector.

The vector $|\psi_e\rangle = |10\rangle$ can also be written as

$$|\psi_e\rangle = 0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle \quad (4.3)$$

This means that the vector $|\psi_e\rangle$ has values $c = 0$, $d = 0$, $e = 1$, and $f = 0$ so that the probability of finding $|\psi_e\rangle$ is 1 in vector $|10\rangle$ and 0 in the other three vectors.

The vector $|\psi_e\rangle$ involves two qubits, $q[0]$ and $q[1]$. Qubit $q[0]$ is $|0\rangle$ and $q[1]$ is $|1\rangle$.

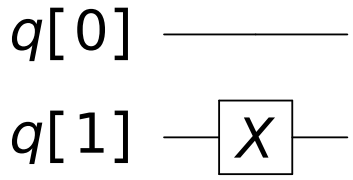

Figure 4.1: Quantum circuit of the vector $|\psi_e\rangle = |10\rangle$

The quantum circuit of the vector $|\psi_e\rangle = |10\rangle$ is presented in Figure 4.1.

In Figure 4.1 there are two qubits, $q[0]$ and $q[1]$. The first qubit $q[0]$ is at the first or top of the circuit, followed by the second qubit $q[1]$.

The vector $|10\rangle$ is the tensor product of the vector $|1\rangle$ and the vector $|0\rangle$.

The tensor product is symbolised as $\otimes$ so that

$$|\psi_e\rangle = |q[1]\rangle \otimes |q[0]\rangle \qquad (4.4)$$

then

$$|\psi_e\rangle = |1\rangle \otimes |0\rangle = |10\rangle \qquad (4.5)$$

Note that in the tensor product operation, the first qubit $q[0]$ is always placed at the backmost or rightmost in the calculation operation.

From (1.3) and (1.4), the vector $|\psi_e\rangle$ can be written in matrix representation as

$$|\psi_e\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad (4.6)$$

◁

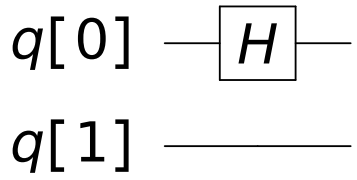

Figure 4.2: Quantum circuit of the vector $|\psi_{0H}\rangle = |0\rangle \otimes H|0\rangle$

▹ Example 4.2.

Given a 2-qubit quantum circuit as presented in Figure 4.2. Tell us about this 2-qubit system.

Figure 4.2 shows that qubit $q[1]$ has value $|0\rangle$ and qubit $q[0]$ has value $H|0\rangle$ thus

$$|\psi_{0H}\rangle = |0\rangle \otimes H|0\rangle \tag{4.7}$$

It has been explained in the previous chapter that $H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ then

(4.7) becomes

$$|\psi_{0H}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad (4.8)$$

or

$$|\psi_{0H}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) \qquad (4.9)$$

The vector $|\psi_{0H}\rangle$ on (4.9) has the value $c = d = \frac{1}{\sqrt{2}}$ and the value $e = f = 0$. This shows the probability of finding $|\psi_{0H}\rangle$ is $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ in vectors $|00\rangle$ and $|01\rangle$ and zero in vectors $|10\rangle$ and $|11\rangle$.

From (1.3) and (3.2), the vector $|\psi_{0H}\rangle$ can also be composed in matrix representa-

tion as

$$|\psi_{0H}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (4.10)$$

◁

• Exercise

4.1. Construct the quantum circuit and the vector equation of the tensor product of $|0\rangle \otimes |0\rangle$. Tell us about the vector.

4.2. Repeat Exercise 4.1. for $H|0\rangle \otimes H|0\rangle$.

5

CX Gate

CX gate, control qubit, target qubit

The *CX* gate, or controlled-*X* gate, is the third quantum gate discussed in this book. Like the *X* gate and the *H* gate, the *CX* gate is also one of the fundamental gates in quantum computing. The *CX* gate is also known as the *CNOT* gate or controlled-*NOT* gate.

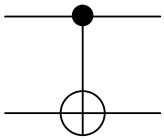

Figure 5.1: Quantum circuit of the *CX* gate

The quantum circuit and the matrix of the *CX* gate are shown in Figure 5.1 and (5.1), respectively.

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.1}$$

Now take a look at the Figure 5.2. Figure 5.2 is the quantum circuit of the vector $CX|01\rangle$. In the figure, the *CX* gate is applied to the vector $|01\rangle$. Qubit $q[0]$ as control qubit and $q[1]$ as target qubit.

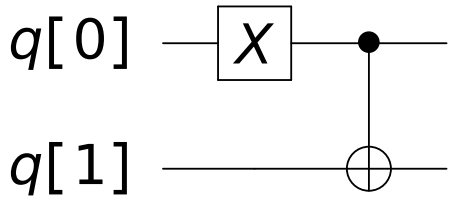

Figure 5.2: Quantum circuit of $CX|01\rangle$

The vector $CX|01\rangle$ can be written as

$$CX|01\rangle = CX(|0\rangle \otimes |1\rangle) \qquad (5.2)$$

then the matrix calculation of the vector $CX|01\rangle$ is

$$CX|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \qquad (5.3)$$

or

$$CX|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (5.4)$$

Thus

$$CX|01\rangle = 0|00\rangle + 0|01\rangle + 0|10\rangle + 1|11\rangle \quad (5.5)$$

or

$$CX|01\rangle = |11\rangle \quad (5.6)$$

It can be said that applying the CX gate to vector $|01\rangle$ will turn it into vector $|11\rangle$.

Notes that the target qubit, qubit $q[1]$, which was originally $|0\rangle$ becomes $|1\rangle$.

- Exercise

5.1. Construct the quantum circuit and calculate the matrix representation of the vector $CX|00\rangle$. Tell us about the second qubit $q[1]$.

5.2. Repeat Exercise 5.1. for the vector $CX|11\rangle$.

6

Entanglement

Separable, not separable, entanglement

All examples of the 2-qubit quantum systems in the previous discussion can be factorised into individual qubits. These quantum systems are called as product states or separable states.

For example in (5.6), the vector $|11\rangle$ can be factorised into a tensor product of the

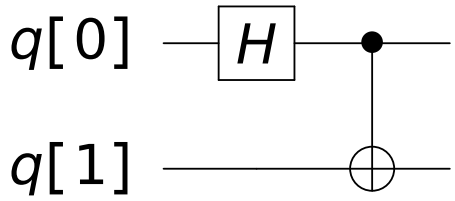

Figure 6.1: Quantum circuit of $|\Phi^+\rangle$

individual qubits of $|1\rangle$ and $|1\rangle$ or written as $|1\rangle \otimes |1\rangle$.

In this chapter will be presented a discussion of the 2-qubit systems that cannot be factorised into individual qubits. The systems are not separable and known as entangled states. Entanglement is the key to quantum computing, along with superposision.

Figure 6.1 shows the quantum circuit of

the state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The calculation of $|\Phi^+\rangle$ is as follows

$$|\Phi^+\rangle = CX(|0\rangle \otimes H|0\rangle) \tag{6.1}$$

then

$$|\Phi^+\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{6.2}$$

so

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.3}$$

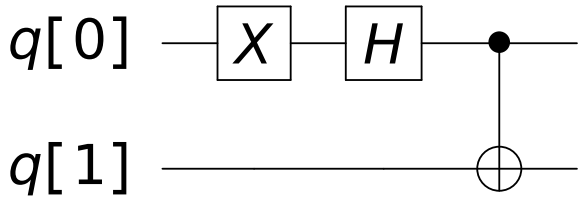

Figure 6.2: Quantum circuit of $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

Thus

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(1|00\rangle + 0|01\rangle + 0|10\rangle + 1|11\rangle) \tag{6.4}$$

or

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{6.5}$$

This state is an entangled state and can not be factorised into individual qubits.

- Exercise

See Figure 6.2 and prove that this quantum circuit has the vector of $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

7

Projects

Project 1 – Consider the quantum circuit of the 2-qubit system in Figure 7.1. Determine the equation of the vector $|\psi\rangle$ and prove that the probability of finding the vector $|\psi\rangle$ is equal to $\frac{1}{2}$ in the vector $|00\rangle$ and the vector $|01\rangle$, respectively.

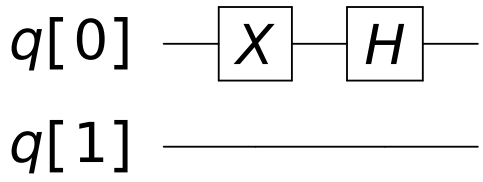

Figure 7.1: Quantum circuit of Project 1

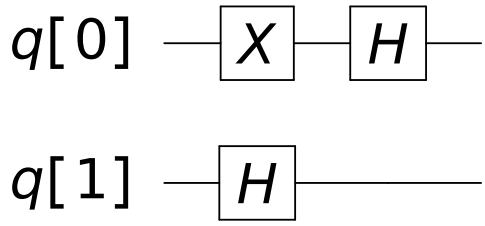

Figure 7.2: Quantum circuit of Project 2

Project 2 – Find the vector equation and the probability of finding the vector $|\phi\rangle$ of the 2-qubit quantum system as shown in the quantum circuit in Figure 7.2.

Project 3 – Take a look at Figure 7.3. The U gate in the figure is known as universal gate. Because its contents are not yet known, it is also called the oracle. Determine the contents of the oracle so that the vector equation of the quantum circuit is $|\varphi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$.

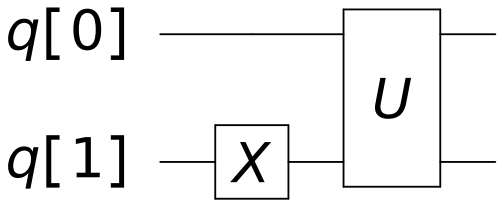

Figure 7.3: Quantum circuit of Project 3

– viel Glück –

Reference

Qiskit-Textbook,

Authors	:	Amira Abbas, *et al.*,
Tittle	:	Learn Quantum Computation using Qiskit
Year	:	2022
URL	:	https://qiskit.org/textbook